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A GEOMETRIC APPROACH TO HYDROTHERMAL SCHEDULING WITH VARIABLE PRODUCTION COEFFICIENT

BERNARDO K. PAGNONCELLI, CARLOS TOMEI, AND HUMBERTO J. BORTOLOSSI

[1,2]Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, RJ, Brasil.
[3]Departamento de Matemática Aplicada, Universidade Federal Fluminense, Niterói, RJ, Brasil.

ABSTRACT. The purpose of these notes is to present computational advances in the analysis of the hydrothermal scheduling problem studied in [2]. In the first part we briefly summarize the main results of the original paper, stating the main theorems without proofs. In the second part we present the computer program wxHSP, designed to obtain numerical solutions to the problem. The numerical routines are described and discussions about the implementation and efficiency of each one is included. We conclude with the application of the geodesic routines to our case study, the El Cajón system in Honduras.

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1. INTRODUCTION

The hydrothermal scheduling problem we will study consists in achieving the least costly operation of a system with one hydro plant and several thermal plants. At each period, a given load has to be fulfilled by the combined use of the hydro and thermal plants. We assume hydro plants are costless.

Hydrothermal scheduling has been studied extensively, both in theoretical and practical terms. Several modelling issues have been addressed: long or short term scheduling; deterministic or stochastic parameters; continuous and discrete time. Another important issue is how to model the hydro generation efficiency. Several authors take it to be constant, simplifying the mathematics involved. We will consider a variable production coefficient that takes into account the geometry of the reservoir.

In the first section we state the discrete version of our problem and discuss modelling issues concerning the production coefficient ρ. In sequence we consider a continuous formulation obtained as the limit of the discrete one, when the number of stages goes to infinity. Several results about the continuous model are stated without proof. The reader is referred to the paper [2] for further details.

We then turn our attention to the computer program wxHSP. The program is a set of routines that combines classical numerical optimization techniques such as gradient methods with procedures based on the theory outlined in the first part of these notes. First we suggest a strategy to find a feasible solution to the problem. We describe which numerical routines were implemented in the program as well as details of the implementation.

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The method inspired by the theory developed in the first section is discussed and the data of our case study, the honduran system of El Cajón is given. We compare the solution obtained by these methods with classical ones and screenshots of the program are displayed to illustrate the exposition.

2. SUMMARY OF PREVIOUS RESULTS

In this section we give a mathematical description of the hydrothermal problem we will consider and briefly state the main theoretical results associated with the model. The details and proofs can be found in the original paper [2].

2.1. **The discrete model.** First we are going to consider a discrete model in the time interval $[0, T]$ with NS stages (see Figure 1). The hydrothermal operation problem consists in determining the values of the outflow q_t and spillage s_t of a hydro plant as well as the generations $g_{t,j}$ of the thermal plants so that the load d_t is satisfied at each stage t with minimal cost. The initial volume v_0 is given and the volume v_t at each stage is updated based on the positive inflow b_t, the outflow q_t and spillage s_t.

FIGURE 1. Discretization of the interval $[0, T]$.

More concretely, we consider the problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{t=1}^{NS} \sum_{j=1}^{NT} c_j(g_{t,j}) \cdot \Delta\tau \\
\text{subject to} & h_t + \sum_{j=1}^{NT} g_{t,j} = d_t, \\
 & h_t = \rho(v_t) \cdot q_t, \\
 & v_{t+1} = v_t + b_t \cdot \Delta\tau - q_t \cdot \Delta\tau - s_t \cdot \Delta\tau, \text{ (hydro balance equation)} \\
 & 0 \le g_{t,j} \le \overline{g}_j, \qquad 0 \le q_t \le \overline{q}, \qquad \underline{v} \le v_{t+1} \le \overline{v}, \qquad 0 \le s_t,
\end{array}
\tag{2.1}
$$

where NT is the number of thermal plants, $\Delta\tau$ is the time length of each stage and $c_j(\cdot)$ is the cost function for thermal plant j. Note that the amount of water in the reservoir is bounded between $\overline{v}$ and $\underline{v}$, as well as the thermal generations (bounded by $\overline{g}_j$) and the outflow (bounded by $\overline{q}$). The hydro generation depends on a *production coefficient* ρ, which is used to account for the shape of the reservoir. For simplicity, we suppose that the thermal plants by themselves are able to generate the requested load.

Many authors consider ρ to be a constant, which is appropriate for hydro plants with large net heads and it also simplifies the mathematics involved in the model. As indicated in the title of this work, we will not use this assumption. A more suitable choice for hydro plants with large water mirrors, as the one in our case study to be studied later in these notes, is to take ρ to be a log-concave function of the volume v_t on the reservoir.

The control variables of the problem are the outflow q_t, the spillage s_t and the thermal generations $g_{t,j}$, for $t = 1, \ldots, NS$ and $j = 1, \ldots, NT$. The cost function will be assumed to be quadratic: $c_j(g_{t,j}) = c_j \cdot g_{t,j}^2$, c_j a positive constant. The production coefficient is usually assumed to be a power function $\rho(x) = c \cdot x^\alpha$, $\alpha > 0$, which is an example of a log-concave function. The major drawback of this choice of ρ is the loss of convexity in the various formulations of the problem.

It is possible to reformulate problem (2.1) in such a way that the only control variables are the outflow q_t and the spillage s_t. For each stage t, assuming a certain fixed value for the hydro generation h_t, we define the *thermal load*:

$$k_t = k_t(v_t, q_t, d_t) = d_t - h_t = d_t - \rho(v_t) \cdot q_t. \tag{2.2}$$

The idea is that since hydro generation does not incur in any costs, we must ask for optimal performance of the thermal plants to meet the thermal load k_t. Omitting the time dependence, the *thermal plants operation problem* is

$$\begin{array}{ll} p(k) = \text{minimize} & \sum_{j=1}^{NT} c_j \cdot g_j^2 \\ \text{subject to} & \sum_{j=1}^{NT} g_j = k, \\ & 0 \leq g_j \leq \overline{g}_j. \end{array} \tag{2.3}$$

Using (2.3), we can reformulate (2.1) using only q_t and s_t as control variables. The new version of the problem will be denoted by M_{NS} and can be written as follows.

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{NS} p(d_t - \rho(v_t) \cdot q_t) \cdot \Delta\tau \\ \text{subject to} & 0 \leq d_t - \rho(v_t) \cdot q_t, \\ & v_{t+1} = v_t + b_t \cdot \Delta\tau - q_t \cdot \Delta\tau - s_t \cdot \Delta\tau \ \text{(hydro balance equation)} \\ & 0 \leq q_t \leq \overline{q} \qquad \underline{v} \leq v_{t+1} \leq \overline{v}, \qquad 0 \leq s_t. \end{array} \tag{2.4}$$

2.2. **The continuous model.** In order to obtain a continuous version of problem (2.4), we let the number of steps NS go to infinity and consider the continuous analogues q, s, b, d defined in the whole interval $[0, T]$: they will be real bounded functions.

$$0 \leq q(t) \leq \overline{q}, \ \ 0 \leq s(t) \leq \overline{s}, \ \ \underline{v} \leq v(t) \leq \overline{v} \text{ and } 0 \leq b(t) \leq \overline{b}, \ \ t \in [0, T]. \tag{2.5}$$

It is convenient to assume that these functions are absolutely continuous, which implies that the fundamental theorem of calculus is valid for all of them. The continuous version M_∞ is written as

$$\begin{array}{ll} \text{minimize} & \int_0^T p(d(t) - \rho(v(t)) \cdot q(t))\, dt \\ \text{subject to} & 0 \leq k_t = d_t - \rho(v_t) q_t, \\ & v'(t) = b(t) - q(t) - s(t) \ \text{(hydro balance equation)} \\ & 0 \leq q(t) \leq \overline{q}, \ \ 0 \leq s(t) \leq \overline{s}, \ \ \underline{v} \leq v(t) \leq \overline{v}. \end{array} \tag{2.6}$$

We call an admissible pair $(q, s) \in (L^\infty([t_0, t_1]))^2$ an *operation* in the interval $[t_0, t_1]$. We do not have convexity in the objective function. Nevertheless, the following result holds.

Theorem 2.1. *The model M_∞ admits an optimal admissible solution.*

The proof can be found in [2].

2.3. **Using the volume as control variable.** Unsurprisingly, spillage should be avoided. The next Theorem shows that we can reduce our search of optimal solution to the so called *thrift operations*:

Theorem 2.2. *The model M_∞ (2.6) admits an optimal solution. Furthermore, it may be taken to be* thrifty, *that is, operations (q, s) for which the induced volume v is an absolutely continuous function and*

1. for $t \in [a, b], v(t) < \overline{v}$ implies $s(t) = 0$ and
2. for $t \in [a, b], v(t) = \overline{v}$ implies $q(t) = \min\{b(t) - v'(t), \overline{q}, d(t)/\rho(v(t))\}$.

Theorem 2.2 allows us to focus only on thrifty operations. They are very natural in the sense that spillage will not occur if the reservoir is not full. More important, the existence of optimal thrifty operations, that is, operations that achieve the minimum of M_∞, leads us to consider a new formulation M_∞^v, where the only control variable is the volume v:

$$\begin{array}{ll} \text{minimize} & \int_0^T p(k(t))\,dt \\ \text{subject to} & 0 \le k(t) = d(t) - \rho(v(t)) \min\left\{ b(t) - v'(t), \overline{q}, \frac{d(t)}{\rho(v(t))} \right\}, \\ & b(t) - v'(t) \ge 0 \\ & \underline{v} \le v(t) \le \overline{v}. \end{array} \tag{2.7}$$

Note that the outflow can be recovered by $q(t) = \min\{b(t) - v'(t), \overline{q}, d(t)/\rho(v(t))\}$ and the spillage by $s(t) = b(t) - v'(t) - q(t)$. Thus, given a feasible operation v one can immediately convert it into a pair of the form (q, s) by using these equations.

In a similar fashion, we can consider the discrete version of formulation (2.7).

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{NS} p(k_t) \\ \text{subject to} & 0 \le k_t = d_t - \rho(v_t) \min\left\{ b_t - \frac{v_{t+1} - v_t}{\Delta\tau}, \overline{q}, \frac{d_t}{\rho(v_t)} \right\}, \\ & \Delta b_t - (v_{t+1} - v_t) \ge 0 \\ & \underline{v} \le v_t \le \overline{v}. \end{array} \tag{2.8}$$

The importance of the discrete version of the hydrothermal problem with only the volume v_t as a control variable will be more clear when we move to the numerical results, specially the Aitken's Double sweep method.

2.4. **Necessary conditions for optimality.** In this section we will characterize local optimality conditions for problem M_∞^v in an interval $[a, b]$. If the reservoir is not full the hydro balance equation simplifies to $v'(t) = b(t) - q(t)$ because there is no spillage. We then take the function $w(t) = v'(t)$ as our control variable instead of $v(t)$. Among the functions $w(t)$ that satisfies

$$\int_a^b w(t)\,dt = v(b) - v(a), \tag{2.9}$$

we want to find the one that minimizes the cost functional

$$F(w) = \int_a^b p\left(d(t) - \rho\left(v(a) + \int_a^t w(\tau)\,d\tau \right) \cdot (-w(t) + b(t)) \right)\,dt. \tag{2.10}$$

The next theorem gives necessary and sufficient conditions for a function w to be a critical point of functional F subject to restriction (2.9).

Theorem 2.3. *A function w satisfying (2.9) is a critical point of functional F defined in (2.10) if and only if*

$$p'(k(t)) = p'(k(a)) \cdot \exp\left(-\int_a^t \frac{\rho'(v(\tau))}{\rho(v(\tau))} \cdot b(\tau) d\tau \right), \tag{2.11}$$

where $k(t) = d(t) - \rho(v(t)) \cdot (-w(t) + b(t))$ is the thermal load at time t.

Theorem 2.3 does not say anything about the critical point. In principle it could be a local minima, a local maxima or a saddle of functional F in the interval $[a, b]$. Surprisingly, a feasible solution of (2.11) is a global minimum of F in $[a, b]$.

Theorem 2.4. *Let the production coefficient ρ be strictly log-concave. If the solution w of (2.11) is admissible in the interval $[a, b]$, then it is the global minimum of M_∞^v in the interval.*

2.5. **A shortest path problem.** Theorem 2.3 suggests a strategy to find solutions for the hydrothermal problem. As long as we restrict ourselves to an interval $[a, b] \subset [0, T]$ and avoid the frontier (where the reservoir is full), the solution of equation (2.11) gives us the optimal operation on the given interval. An operation can be seen as a path in a $t \times v(t)$ space, so we can relate our problem to a shortest path problem.

Consider an open set $\mathcal{R}$ of the plane bounded by a simple curve Γ and let A and Z be points in the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$. We want to find the shortest path between A and Z in $\overline{R}$. One strategy is to apply repeatedly the following Proposition, whose proof is left to the reader.

Proposition 2.5. Let r be a straight line segment in $\mathcal{R}$. Let $\mathcal{R} - r$ split into open path connected components $\mathcal{R}_\alpha, \alpha = 1, 2, \ldots$. Let $\mathcal{R}_A$ and $\mathcal{R}_Z$ be components having A and Z in their closures, respectively. If the component $\mathcal{R}_{\alpha_0}$ does not meet $\mathcal{R}_A$ and $\mathcal{R}_Z$, the shortest path does not intersect it.

As an example, consider Figure 2 where we show the shortest between points A and Z. The shaded regions generated by the line segments r_1 and r_2 were ruled out using Proposition 2.5.

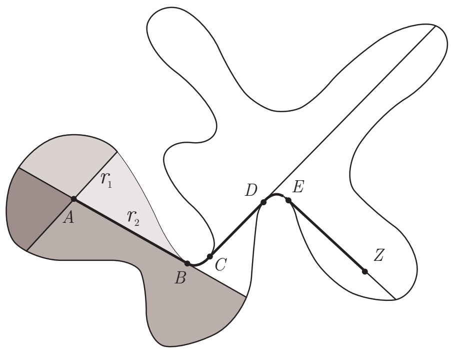

FIGURE 2. The shortest path between A and Z.

The hydrothermal problem (2.7) with the volume as a control variable can be seen as a shortest path problem. Unlike the Euclidean case, the shortest path between points is given by Equation (2.11), as long as the solution remains feasible. There are additional restrictions that may interrupt the validity of a solution: The bounds $b(t) - \overline{q} \leq v(t) \leq b(t)$. Fortunately, the optimal performance in our case study was far from this situation.

2.6. **Discrete geodesics.** We derived the optimality conditions for the continuous version of the problem on Theorem 2.3. In a similar fashion, we can obtain the optimality conditions for the discrete problem (2.8), where the only control variable is v_t. More precisely, consider an interval $[a, b] \subset [0, T]$ and a discretization in NS subintervals of equal size. A critical point $(v_2, \ldots, v_{NS+1})$ of the functional

$$F(v_2, \ldots, v_{NS+1}) = \sum_{t=1}^{NS} p\left(d_t - \rho(v_t) \cdot \left(-\frac{v_{t+1} - v_t}{\Delta\tau} + b_t\right)\right) \cdot \Delta\tau \tag{2.12}$$

must satisfy

$$\frac{\partial F}{\partial v_2} = \cdots = \frac{\partial F}{\partial v_{NS+1}} = 0. \tag{2.13}$$

Rearranging equations (2.13), we have that a critical point $(v_1, \ldots, v_{NS+1})$ of (2.12) must satisfy for $t = 2, \ldots, NS$

$$p'(k_{t-1}) \cdot \rho(v_{t-1}) - p'(k_t) \cdot \rho(v_t) = p'(k_t) \cdot \rho'(v_t) \cdot \left(-\frac{v_{t+1} - v_t}{\Delta\tau} + b_t\right), \tag{2.14}$$

where

$$k_t = d_t - \rho(v_t) \cdot \left(-\frac{v_{t+1} - v_t}{\Delta\tau} \right). \tag{2.15}$$

Summing over $t = 2, \ldots, NS$ we obtain

$$p'(k_1) \cdot \rho(v_1) - p'(k_{NS}) \cdot \rho(v_{NS}) = -p'(k_1) \cdot \rho'(v_t) \cdot \left(-\frac{v_2 - v_1}{\Delta\tau} + b_1 \right) + \tag{2.16}$$

$$\sum_{t=1}^{NS} p'(k_t) \cdot \rho'(v_t) \cdot \left(-\frac{v_{t+1} - v_t}{\Delta\tau} + b_t \right). \tag{2.17}$$

Letting $NS \to \infty$, or equivalently $\Delta\tau \to 0$, we have

$$p'(k(a)) \cdot \rho(v(a)) - p'(k(b)) \cdot \rho(v(b)) = \int_a^b p'(k(\tau)) \cdot \rho'(v(\tau)) \cdot (-v'(\tau) + b(\tau)) d\tau, \tag{2.18}$$

which is equal to equation (2.11) up to a change of variables. Thus, we conclude that the limit of discrete solutions in $[a, b]$ of equations (2.13) (discrete geodesics) is the continuous geodesic in the given interval.

3. wxHSP: A NUMERICAL/GRAPHICAL FRAMEWORK FOR THE HYDROTHERMAL SCHEDULING PROGRAM

The program wxHSP is written in C++ using the object-oriented programming features to model the hydrothermal system. It has a graphical user interface built with the free multi-platform GUI library wxWidgets[1]. For this reason one can run the program in any operational system. A binary version for Microsoft Windows© can be downloaded from the address

`http://www.mat.puc-rio.br/~hjbortol/wxHSP/wxHSP.zip`.

The Figure 3 shows a screenshot of the program. The window Ⓐ is the main control panel where the user can apply several numerical methods, load and save files and dump data for analysis. We now will describe the numerical methods available in wxHSP. All algorithms use the volumes as control variables.

3.1. Obtaining a feasible operation.

To start using the program we must compute an initial feasible operation. This is not a difficulty task under the hypothesis that the thermal plants are able to fulfill the total load by themselves. Indeed, a feasible operation $v_t^{[d]}$ can be computed defining $v_1^{[d]} =$ initial volume and, then, recursively,

$$v_t^{[d]} = \max \left\{ v_{t-1}^{[d]} + b_t \cdot \Delta\tau - \min \left\{ \overline{q}, \frac{d_{t-1}}{\rho(v_{t-1}^{[d]})} \right\} \cdot \Delta\tau,\ \underline{v} \right\}, \quad \text{for } t = 2, \ldots, NS.$$

In this operation, all water available is readily used: the hydro plant operates under its maximal outflow ($\overline{q}$) or it generates all load being demanded, observing, of course, the restriction $v_t^{[d]} \geq \underline{v}$. However, this approach is too greedy in the sense that it does not worry about lack of water in the future. Consequently, we may end up with high costs at later periods of the planning horizon, since a large percentage of the load will have to be fulfilled almost exclusively by the thermal plants.

On the other hand, we may take a totally different approach where we always choose to save water for the future using zero outflow, unless the reservoir is full. From this moment on, the hydro plant operates using the outflow necessary to keep the reservoir full. Note that spillage may occur in this case. This defines the admissible operation $v_t^{[f]}$, where $v_1^{[f]} =$ initial volume and

$$v_t^{[f]} = \begin{cases} v_{t-1}^{[f]} + b\tau \cdot \Delta\tau, & \text{if } v_{t-1}^{[f]} + b\tau \cdot \Delta\tau < \overline{v}, \\ \overline{v}, & \text{otherwise.} \end{cases}$$

[1] `http://www.wxwidgets.org`.

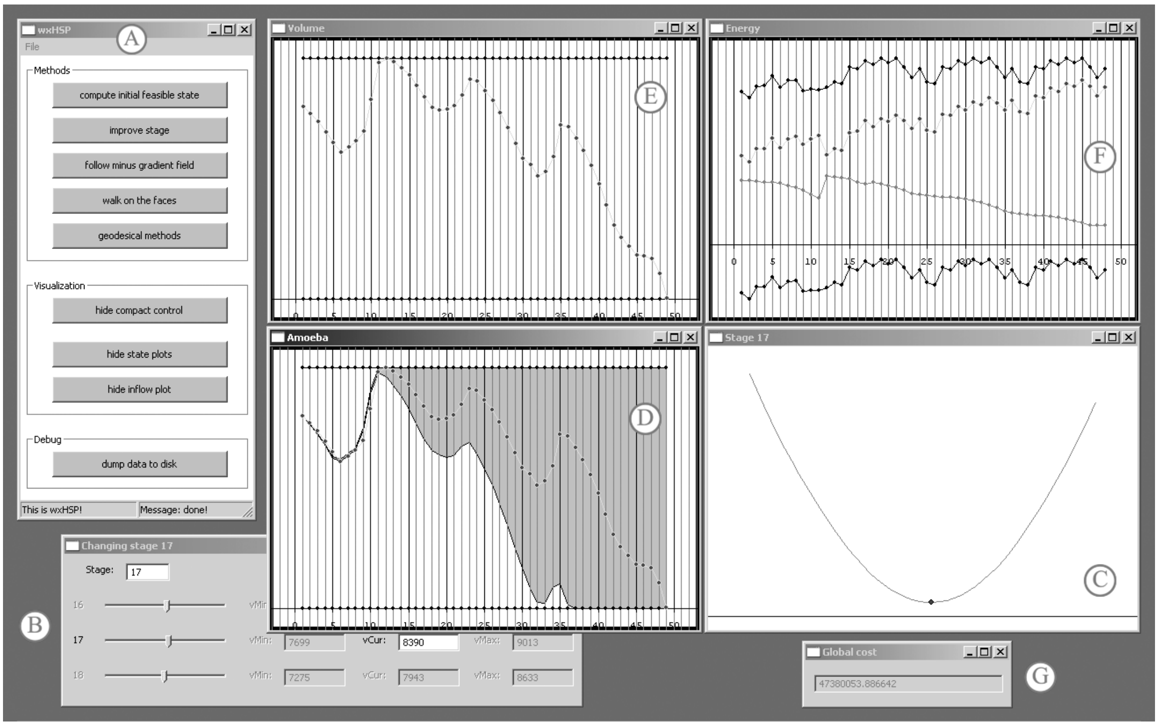

FIGURE 3. Program screenshot.

Actually, the strategy implemented in the program wxHSP uses a weighted combination of these two feasible operations, where the weight is selected employing a bisection method. More precisely, we define $\lambda_B = 0$ and $\lambda_E = 1$ and, then, we repeat the following steps a certain number of times: (1) Define $\lambda_M = (\lambda_B + \lambda_E)/2$. (2) Define $v_t = (1 - \lambda_M) \cdot v_t^{[f]} + \lambda_M \cdot v_t^{[d]}$ for $t = 1, \ldots, NS + 1$. (3) If the operation v_t is feasible, we set $\lambda_B = \lambda_M$, otherwise, we set $\lambda_E = \lambda_M$.

3.2. **An unidimensional sequential method based on the Aitken's double sweep scheme.** Taking the reservoir's volumes as control variables has a great advantage: changes in a single control variable v_t (for $2 \leq t \leq NS$) affects *only* two terms of the total cost, namely,

$$p\left(d_{t-1} - \rho(v_{t-1}) \cdot \min\left\{b_{t-1} - \frac{v_t - v_{t-1}}{\Delta\tau}, \overline{q}, \frac{d_{t-1}}{\rho(v_{t-1})}\right\}\right)\Delta\tau + p\left(d_t - \rho(v_t) \cdot \min\left\{b_t - \frac{v_{t+1} - v_t}{\Delta\tau}, \overline{q}, \frac{d_t}{\rho(v_t)}\right\}\right)\Delta\tau. \tag{3.1}$$

So, for fixed values of v_{t-1} and v_{t+1}, we may compute the optimal v_t that minimizes the expression above. This is done in wxHSP using the univariate Brent's method plus a smart handling of the combinatorics in the computation of the two "$\min$". Now, this procedure can be applied sequentially to improve a given feasible operation: we start improving v_2 for fixed values of v_1 and v_3, next we improve v_3 for fixed values of v_2 and v_4, and so on. When we do a round trip (from v_2 to v_{NS-1} and, then, back to v_2), we get the double sweep scheme suggested by Aitken. This algorithm is accessible in wxHSP through the button "improve stage" in Ⓐ.

Also, for fixed values of v_{t-1} and v_{t+1}, the user can adjust the value of v_t manually using the dialog window Ⓑ. The graph of the partial cost (expression (3.1)) as function of v_t is shown in the window Ⓒ.

3.3. **Classical methods.** Two classical strategies in optimization are implemented in the program: "follow the gradient field" and "walk on the faces" buttons in Ⓐ. The first basically chooses variables that are not active at any constraint and follow the direction of minus the gradient vector of the cost function until (at least) one constraint becomes active. The second approach searches for active constraints and do a search in the "interior" of the constraint, trying to improve the current solution. Since those two approaches are classic, we give no further details and refer the reader to [4]. It is worth noting that, for the numerical experiments realized so far, neither strategy provided substantial decrease in the overall cost.

3.4. **Continuous geodesics.** Continuous geodesics are constructed using the integro-differential equation (2.11). To solve it numerically we employed a method analog to Heun's method ([3]) (with quadratic convergence). Methods with higher convergence, similar to Runge-Kutta for differential equations, could have been used but the simpler method served well for our purposes. The integral

$$\int_a^t \frac{\rho'(v(\tau))}{\rho(v(\tau))} \cdot b(\tau) d\tau$$

was estimated with the trapezoid rule. The numerical computation of a continuous geodesic is available through the button "geodesical methods" in Ⓐ. When a geodesic produces a cut, this information is registered in the window Ⓓ.

3.5. **Discrete geodesics.** In section 2.6, we derive an expression (equation 2.14) for the critical points of the functional F. From equation (2.14), we see that given v_{t-1} and v_t, it is possible to obtain v_{t+1}. Furthermore, since $p(k)$ defined in (2.3) is quadratic by parts, Equation (2.14) is quadratic by parts, that is, in order to find v_{t+1} one must find the zeroes of a quadratic by parts equation.

Thus, $\partial F/\partial v_t = 0$ constitute a "triangular" nonlinear system: given v_{t-1} and v_t, one can immediately use equation $\partial F/\partial v_t = 0$ to obtain v_{t+1}. That is how the routine works: the user gives initial data v_0 and v_1 and the program calculates a sequence of volumes that solve the system (2.13). The routine terminates when a feasible cut is obtained, when an infeasible operation is generated or when a quadratic equation has no solution.

There is no particular reason to believe that discrete geodesics can help in any way. On section 2.6, we proved a convergence result for discrete geodesics so one expects that discrete geodesic cuts may be of some help. In fact we can see on the program that discrete geodesic cuts do provide meaningful insight on the optimal solution by ruling out regions of the state space.

3.6. **Computing a global solution.** Theorem 2.2 guarantees the existence of an optimal solution. We know by Theorem 2.4 that solutions of equation (2.11) are global solutions to the problem, as long as they remain feasible. We do not have an algorithm in the sense of a list of steps predetermined, but the heuristics is to draw geodesic cuts in the space $[0, T] \times v(t)$ to at least reduce the feasible region. We applied this methodology to our case study, the El Cajón system in Honduras.

The El Cajón system in Honduras consists of one hydro plant and eight thermal plants. The numerical experiments were performed for a period of 4 years, broken onto $NS = 48$ months. The inflow b_t and the load d_t are given for every stage. The initial volume of the reservoir is $v_0 = 4.85898\,10^9\text{m}^3$, the volume bounds are $\underline{v} = 1.6841\,10^9\text{m}^3, \overline{v} = 5.62527\,10^9\text{m}^3$ and $\overline{q} = 204.8\text{m}^3/s$. The production coefficient was taken to be $\rho(x) = 2.17576\,10^{-5}\sqrt{x}\,\text{MW}/\text{m}^3/\text{s}$.

Using standard optimization techniques (Aitken double sweep method, steepest descent and projected gradient evolutions on faces), an admissible operation was obtained, without spillages. The heuristics using the cuts reduced the feasible region significantly. In window Ⓓ of Figure 3, the discrete geodesic cuts eliminated all the withe region, reducing our search for an optimal solution to the much smaller gray region. The solution obtained with was 5% better than the one obtained with classical optimization techniques.

In the wxHSP, the user can access this method by clicking on the button "geodesic methods" and then selecting the (default) option "continuous geodesic". The user can apply the method between any desired stages by filling the "from/to stage" field. The step size and tolerance of the method can also be adjusted, as well as the initial volume and derivative of $v(t)$, which are the initial conditions for the numerical simulation.

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