Random linear systems and simulation

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Abstract. In this paper we study the well-known linear system of differential equations in the plane. Instead of considering a fixed matrix A, we assume that its entries are random variables and we argue about the probability of the origin be each of the fixed points, such as saddle, node, etc. The problem was suggested by Steven Strogatz in his book [8]. We examine the problem not only from the theoretical viewpoint but also from the computational side, using simulation to obtain approximate answers.
Keywords: ***system of differential equations. random matrices. simulation.***

1 Introduction

Phenomena involving continuous evolution in time are often modeled using differential equations. The simplest and most used models are built using linear differential equations with constant coefficients. Those equations can be easily reduced to systems of linear differential equations of first order. The examples range from a simple harmonic oscillator [8] to an oligopoly model where a number of firms compete for the market [7].

Precisely, we will consider two-dimensional homogeneous linear systems with constant coefficients a,b,c,d:

$$\begin{array}{rcl} \dot{x} & = & ax + by \\ \dot{y} & = & cx + dy \end{array},$$

where $x = x(t)$, $y = y(t)$, t is an independent variable and the dot denotes derivative. In matrix form we have

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Observe that $(0, 0)$ is always a fixed point of this system. On basic courses of differential equations one studies the possible configurations of the phase space associated with this system. Different situations occur according to the signs of the trace τ of A, the determinant Δ of A and of $\tau^2 - 4\Delta$. We say that a fixed point is a *saddle* if $\Delta < 0$ (figure 1). If $\Delta > 0$, $\tau < 0$ (respectively $\tau > 0$) and $\tau^2 - 4\Delta > 0$ we have a *stable* (respectively *unstable*) *node* (figure 2). If $\Delta > 0$, $\tau < 0$ (resp. $\tau > 0$) and $\tau^2 - 4\Delta < 0$ we have a *stable* (respectively *unstable*) spiral (figure 3) and if $\tau = 0$ we have a *center* (figure 6). Finally, there are the borderline cases: if $\tau^2 - 4\Delta = 0$ we have a degenerate node or a star (figures 4 and 5), depending on the number of eigenvectors (1 or 2) associated with the eigenvalue of A. If $\Delta = 0$ we have non-isolated fixed points.

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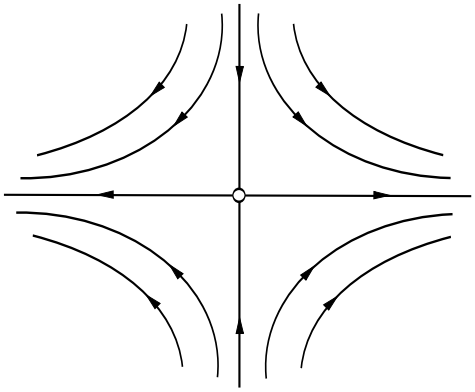

Figure 1: saddle

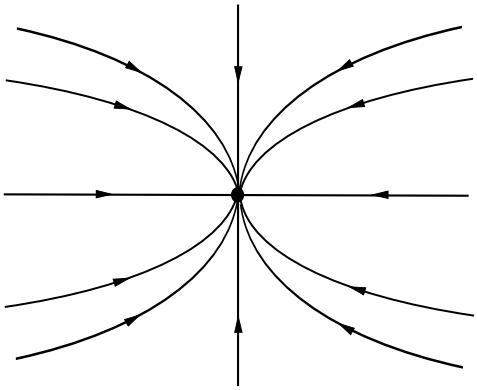

Figure 2: stable node

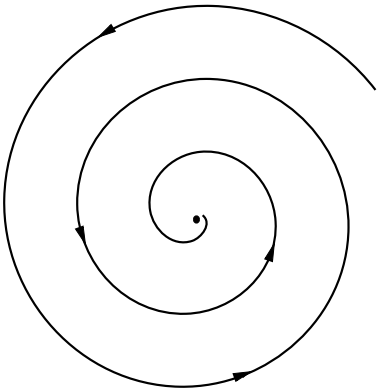

Figure 3: stable spiral

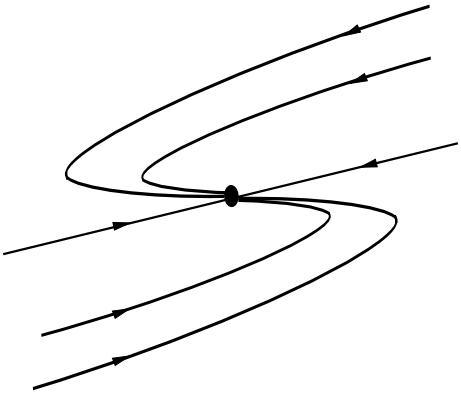

Figure 4: degenerate node

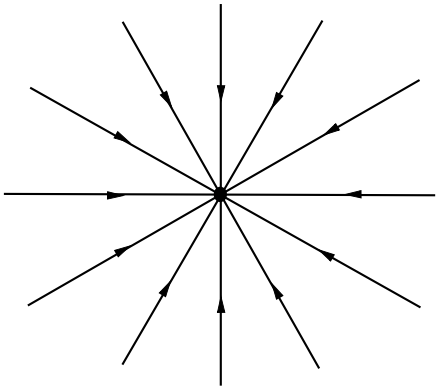

Figure 5: star

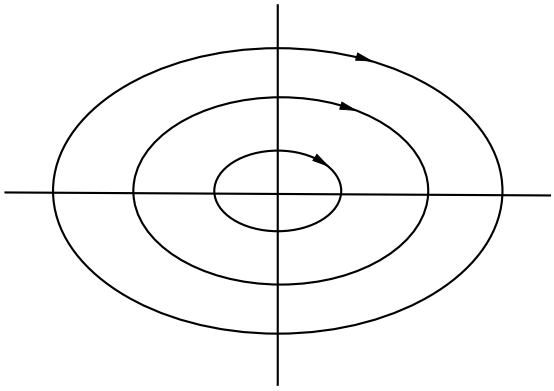

Figure 6: center

2 Random Matrices

The main purpose of this paper is to consider the matrix A that describes a two-dimensional linear system with random entries, or a random matrix. More precisely, we say that a matrix A is a *random matrix* when its entries are random variables (r.v.) with a specified distribution. The problem we will study originated from an exercise of the book "Nonlinear Dynamics and Chaos", by Steven H. Strogatz [8]. Suppose we randomly pick a linear system. What's the probability that the origin is,

for instance, a stable spiral?

Consider the matrix A that describes a two-dimensional linear system with entries X_1, X_2, X_3 e X_4, where each entry is a r.v.:

$$A = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

In this paper we will consider r.v. that are independent and identically distributed (i.i.d.). Randomly picking a linear system means that one randomly picks from the r.v. $X_i,\ i = 1, \ldots, 4$. Denoting by lower case the outcomes of the r.v. we have the matrix

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

which can easily be classified as one of the critical points. Note that some kinds of fixed point have zero probability of occurrence if the r.v. is continuous. Take the center for example, which is characterized by having $\tau = X_1 + X_4 = 0$. We have

$$\mathbb{P}((0,0) \text{ be a center}) = \mathbb{P}(\tau = 0) = 0$$

The same holds for degenerate nodes and stars, which appear when $\tau^2 = 4\Delta$.

3 The random variables $U[-1, 1]$ and $\eta(0, 1)$

We will consider the cases where the r.v. are uniformly distributed in the interval $[-1, 1]$, denoted by $U[-1, 1]$ and normally distributed with mean 0 and variance 1, denoted by $\eta(0, 1)$. In both cases we have symmetric r.v. in the sense that $\mathbb{P}(X < 0) = \mathbb{P}(X > 0)$. This choice aims to give no privilege to any kind of fixed points. If we had chosen uniform r.v. with values between 0 and 1 we would have only positive traces and for instance $\mathbb{P}((0, 0) \text{ be a stable spiral}) = 0$. The r.v. $U[-1, 1]$ has density function given by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad -1 \leq x \leq 1 \\ 0, & \text{otherwise} \quad . \end{cases}$$

The density of a r.v. normally distributed $\eta(0, 1)$ is

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ ,\ -\infty \leq x \leq \infty.$$

The graphs of these density functions are shown below.

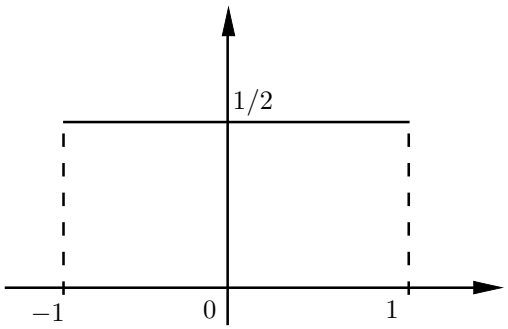

Figure 7: $U(-1, 1)$

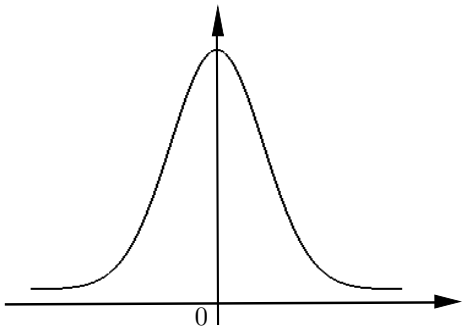

Figure 8: $\eta(0, 1)$

The random matrix with entries $\eta(0, 1)$ appears frequently in the literature. This is not by chance: the normal r.v. has some desired properties that make its choice natural. In order to provide an example, notice that asking whether a 2×2 matrix A is a stable spiral is the same as asking if $200A$ is a stable spiral. The problem is invariant to positive scalar multiplication. This fact implies that trace, determinant and $\tau^2 - 4\Delta$ have the same signs for A and $200A$.

Using this invariance we can provide a geometric interpretation to the original problem posed in section 2. Normalizing the matrices so that they all have norm equal to 1, we can think that instead of $\mathbb{R}^4$ we are calculating probabilities in $S^3 \subset \mathbb{R}^4$. Observe that if we had allowed multiplication by an arbitrary scalar we should consider the projective space $\mathbb{P}^3$. The invariance of multiplication by positive scalars also tells us that the regions in $\mathbb{R}^4$ corresponding to each type of behavior of the origin are cones: for example, if (x_0, y_0, z_0, w_0) represents a matrix that is a stable node than the projection of this point in S^3 represents the whole line $\alpha(x_0, y_0, z_0, w_0)$, $\alpha \in \mathbb{R}^+$. We could ask ourselves what is the probability distribution in the sphere. The answer is quite simple: it is uniform because for $(x_1, x_2, x_3, x_4) \in S^3$ we have,

$$g(x_1, x_2, x_3, x_4) = f(x_1)f(x_2)f(x_3)f(x_4) = \frac{1}{\sqrt{2\pi}^4} e^{(x_1^2+x_2^2+x_3^2+x_4^2)} = K,$$

where K is a constant equal to $\frac{1}{\sqrt{2\pi}^4}$ e since $(x_1, \ldots, x_4)$ belongs to S^3 and $g(x_1, x_2, x_3, x_4)$ is the joint density of $X_i, i = 1, \ldots, 4$. This result obviously is valid for spheres of any radius.

What we have shown is that a normal independent distribution in $\mathbb{R}^4$ implies an uniform distribution in S^3. What about the reciprocal of this fact? Which independent marginal densities should we consider in order to obtain a joint density function constant on spheres? To save space, let's run the calculations for 2 dimensions. We want to find $f_1(x)$ e $f_2(y)$ such that

$$f_{X,Y}(x, y) = f_1(x)f_2(y) = h(x^2 + y^2),$$

where $f_{X,Y}(x, y)$ is a two-dimensional density and $f_1(x)$ and $f_2(y)$ are one-dimensional densities. Assuming that f, g and h have at least one derivative we have

$$f_1'(x)f_2(y) = 2xh'(x^2 + y^2) \text{ and } f_1(x)f_2'(y) = 2yh'(x^2 + y^2).$$

Equating both equations and assuming x and y different of zero gives us

$$\frac{f_1'(x)}{xf_1(x)} = \frac{f_2'(y)}{yf_2(y)} = \lambda,$$

where λ is an arbitrary constant because each term depends only on one of the variables. Solving the differential equation we obtain expressions for the one-dimensional densities:

$$f_1(x) = C_1 e^{\frac{\lambda x^2}{2}} \text{ and } f_2(y) = C_2 e^{\frac{\lambda y^2}{2}},$$

which are normal densities with zero mean variance depending on λ. Thus, if we want a joint density function that is constant in spheres it suffices to choose one-dimensional normal r.v. for each direction.

4 Searching for a solution

For one of the fixed points, namely the saddle, it is possible to obtain explicitly the probability of occurrence in a simple way. We will make one assumption:

$$\mathbb{P}(X_i > 0) = \mathbb{P}(X_i < 0) = 1/2, \; i = 1, \ldots, 4. \tag{1}$$

The two r.v. we are considering satisfy this condition. Remember that the origin is a saddle if and only if

$$\mathbb{P}((0,0) \text{ be a saddle}) = \mathbb{P}(\Delta < 0) = \mathbb{P}(X_1X_4 - X_2X_3 < 0) = P(Y_1 < Y_2),$$

where $Y_1 = X_1X_4$ and $Y_2 = X_2X_3$. By construction Y_1 and Y_2 are identically distributed. Besides, they inherit the property described in 1:

$$\mathbb{P}(Y_1 > 0) = \mathbb{P}(X_1 > 0 \, , \; X_4 > 0) + \mathbb{P}(X_1 < 0 \, , \; X_4 < 0) = \frac{1}{2}\,\frac{1}{2} + \frac{1}{2}\,\frac{1}{2} = \frac{1}{2}.$$

Note that in the second passage we used that X_i, $i = 1, \ldots, 4$ are independent. The calculation above implies that $\mathbb{P}(Y_1 < 0) = 1/2$ by complementarity. The same holds for Y_2. Our problem now is to calculate $\mathbb{P}(Y_1 < Y_2)$. The space of possible outcomes of (Y_1, Y_2) is $\Omega_{Y_1,Y_2} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in \Omega_Y \text{ e } y_2 \in \Omega_Y\}$, where Ω_Y is the common sample space of both Y_1 and Y_2. We want to know what is the probability of the pair (Y_1, Y_2) belongs to the set $T = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 < y_2\}$. Figure 9 exemplifies the set T for the $U[-1, 1]$ case:

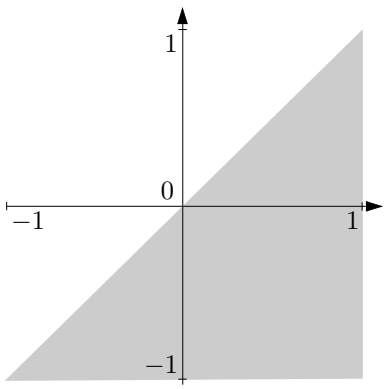

Figure 9: set T

Having picture 9 in mind it is easy to calculate the probability of the origin be a saddle:

$$\begin{aligned}
&\mathbb{P}((0,0) \text{ be a saddle}) = \mathbb{P}(Y_1 < Y_2) = \mathbb{P}((Y_1, Y_2) \in T) \\
&= \frac{1}{2}\mathbb{P}(Y_1 > 0, Y_2 > 0) + \mathbb{P}(Y_1 > 0, Y_2 < 0) + \frac{1}{2}\mathbb{P}(Y_1 < 0, Y_2 < 0) \\
&= \frac{1}{2}\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\frac{1}{4} = \frac{1}{2}
\end{aligned}$$

We will now analyze the unstable spiral. We saw in section 1 that the origin is an unstable spiral if $\tau > 0$, $\Delta > 0$ e $\tau^2 - 4\Delta < 0$. So, the probability that the origin is an unstable spiral is

$$\begin{aligned}
&\mathbb{P}((0,0) \text{ be an unstable spiral}) \\
&= \mathbb{P}(\tau > 0\ ,\ \Delta > 0\ ,\ \tau^2 - 4\Delta < 0) \\
&= \int\int\int\int_D f(x_1, x_2, x_3, x_4)\, dx_1\, dx_2\, dx_3\, dx_4 \\
&= \int\int\int\int_D f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) f_{X_4}(x_4)\, dx_1\, dx_2\, dx_3\, dx_4,
\end{aligned}$$

where $f_{X_i}(x_i)$ are the densities of the independent random variables $X_i,\ i = 1, ..., 4$, $f(x_1, x_2, x_3, x_4)$ is the joint probability density of X_1, X_2, X_3, X_4 and $D = \{(x_1, x_2, x_3, x_4) \in \mathbb{R} \mid x_1 + x_4 > 0,\ x_1x_4 - x_2x_3 > 0, (x_1 + x_4)^2 - 4(x_1x_4 - x_2x_3) < 0\}$.

The integral above is not simple to solve because it is not easy to explicit the limits of integration. In fact, on calculus courses multiple integrals are calculated using Fubini's theorem, which change them into iterated integrals. Let $g(x)$ and $h(x)$ be functions of one variable such that for all x we have $g(x) < h(x)$ and R be the region delimitated by the lines $x = a,\ x = b$ and by the graphs $y = g(x)$ and $y = h(x)$. Suppose we want to calculate the double integral of a function $f(x, y)$ in R. Then we would have

$$\int\int_R f(x,y)dxdy = \int_a^b \int_{g(x)}^{h(x)} f(x,y)dxdy.$$

In the general case what is often done is to divide the region and/or to change variables to fall into this case. Strange as it may seem, this procedure is quite limited. For example, for a region defined by $g(x, y) < 0,\ h(x, y) < 0$ with $g(x, y)$ and $h(x, y)$ polynomials of degree 2, the integral may not be tractable, as for a region defined by $g(x, y) < 0$ with $g(x, y)$ polynomial of degree 5. For this reason numerical methods are frequently employed to deal with integrals.

Before entering next section we would like to make a warning. One may have the temptation of solving this problem in the case $U[-1, 1]$ by simply calculating the corresponding area of each of the critical points in the so called *stability diagram*, drawn in the figure 10.

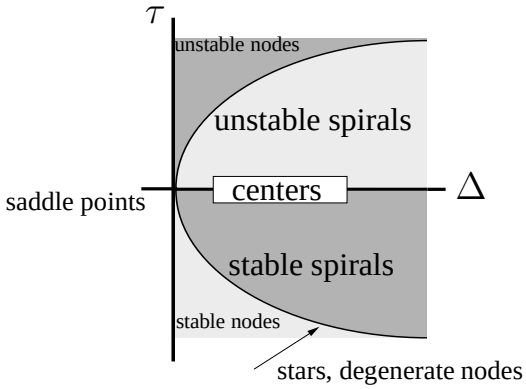

Figure 10: stability diagram

Note that the rectangle $[-2,2]\times[-2,2]$ represents all possible traces and determinants obtained by randomly picking the entries of matrix A according to a uniform distribution between -1 and 1. Except for the saddle node, the quotients of each of the areas divided by the area of the rectangle does not correspond to the probability of obtaining each critical point. This happens because the axis of the figure are τ and Δ and *they are not uniform random variables in* $[-1,1]$. Let's take a closer look at the trace: it is the sum X_1+X_4 of two $U[-1,1]$ r.v. What is the cumulative distribution of this sum? The answer is

$$\mathbb{P}(\tau\leq x)=\begin{cases}(\frac{x+2}{2})^2 & \text{if} \quad -2<x<0,\\ 2(\frac{x+2}{2})-\frac{(\frac{x+1}{2})^2}{2}-1 & \text{if} \quad 0\leq x\leq 2.\end{cases}$$

A reference that helps to obtain the above formula is in [4]. To obtain a formula for the cumulative distribution of the sum of n random variables uniformly distributed check [1], freely available at http://cgm.cs.mcgill.ca/ luc/rnbookindex.html.

5 Monte Carlo method

Monte-Carlo method is a largely used tool for obtaining approximations of problems that are hard to solve explicitly. It consists of generating random numbers and keeping track of the fraction of them that satisfies a certain property. In the context of our problem we will use Monte-Carlo to estimate the probabilities of $(0,0)$ be each of the fixed points. The implementation is simple and can be done following the steps described below:

Initializing the variables Initialize variables saddle, unstable node, stable node, unstable spiral and stable spiral as zero and define variables tau for trace, delta for determinant and tauDelta for $\tau^2-4\Delta$. Proceed as follows:

Step 1 Generate X_1,X_2,X_3,X_4 with distribution $U[-1,1])$ or $\eta(0,1)$.

Step 2 set tau = X_1+X_4 , Delta = $X_1X_4-X_2X_3$ and tauDelta = $\tau^2-4\Delta$.

Step 3 if delta <0 set saddle = saddle $+1$ and return to step 1.

Step 4 if tauDelta >0 and tau >0 (respectively tau <0) set unstable node $=$ unstable node $+1$ (respectively stable node $=$ stable node $+1$) and return to step 1.

Step 5 if tau >0 (respectively tau <0) set unstable spiral $=$ unstable spiral $+1$ (respectively stable spiral $=$ stable spiral $+1$) and return to step 1.

The estimate for each of the probabilities is obtained dividing one of the variables (e.g. stable node) by n. We wrote a code in *Java* and ran the program for various values of n. *Java* language provides a very useful package called Random to deal with the generation of random numbers. For example in this package it is possible to generate a random number U between 0 and 1. The routine described above was run for both $U[-1,1]$ and $\eta(0,1)$.

Before analyzing the results obtained we will discuss an important issue concerning Monte-Carlo estimators: the variance reduction techniques. The goal of these techniques is to avoid an excessive number of generations of the random variable, reducing the sample variance of the Monte-Carlo estimator $\bar{X}$. To provide an example we will consider the simplest variance reduction technique, called the method of the antithetic variables. The idea is quite simple: instead of sampling the r.v. sequentially one organizes the sample in pairs of r.v negatively correlated. More precisely: if the r.v. being sampled is X,

we choose a function $f(X)$ such that X and $f(X)$ have the same distribution but are negatively correlated. Thus, instead of considering outcomes $\{X_i\}_{i=1}^n$ we will consider $\{W_i\}$, where $W_i = (X_i + f(X_i))/2$.

A possible way of detecting the improvement of this technique is to execute the program until the sample standard deviation $S/\sqrt{n}$, where $S^2 = \frac{\sum(X_i-\bar{X})^2}{n-1}$, is less than or equal to a pre-fixed value d and at least 100 simulations were performed. It is expected that a smaller sample is needed to achieve the desired level of confiability. More details can be found in [6].

As expected, we obtained some gain using antithetic variables. Tables (1) and (2) show us the gain in using this method in the case of the stable spiral. The random variable considered was $\eta(0,1)$.

d	number of simulations
1	100
0.1	100
0.01	1156
0.001	124187
0.0001	12499179

Table 1: Monte-Carlo simulation

d	number of simulations
1	100
0.1	100
0.01	984
0.001	103740
0.0001	10357100

Table 2: Monte-Carlo simulation with antithetic variables

6 Results

Initially we performed 100.000 experiments for each distribution on a Pentium 4 with 2.00GHz. Remember that in each experiment four random numbers were generated, representing the entries of the random matrix being considered. The results are shown in table 3:

	$U[-1,1]$	$\eta(0,1)$
saddle	0.50105	0.49931
stable node	0.08881	0.10351
unstable node	0.08929	0.10351
stable spiral	0.15916	0.14563
unstable spiral	0.16169	0.14819

Table 3: Monte-carlo simulation for 100.000 experiments

The execution time was less than one second in both cases. We also run the program for $1.000.000$ experiments and the results are in table 4.

The execution times were less than one second for the $U[-1,1]$ and about 3 seconds for the $\eta(0,1)$. Note that the approximated probability for the saddle is in accordance with the exact value $1/2$ derived in section 4.

	$U[-1,1]$	$\eta(0,1)$
saddle	0.50036	0.49961
stable node	0.09054	0.10347
unstable node	0.09036	0.10336
stable spiral	0.15963	0.14656
unstable spiral	0.15910	0.14699

Table 4: Monte-Carlo simulation for $1.000.000$ experiments

7 Conclusions and extensions

Random matrices appear in many different contexts. Given the computational power we have available nowadays it is almost unavoidable to use simulation to obtain approximations and/or empirical evidence of problems involving random matrices. Besides the problem considered here, there are many intriguing questions related to random matrices: given a $n \times n$ random matrix A with entries distributed according to a $\eta(0,1)$ distribution, what is the probability that exactly k of its eigenvalues are real? What is the expected number of real eigenvalues? After normalized to lie in the $[-1,1]$ interval, what is the distribution of the real eigenvalues of A? A known result is that if the r.v. are $\eta(0,1)$, the probability that all eigenvalues of a $n \times n$ random matrix A are real is $1/2^{n(n-1)/4}$. Using this result we can obtain an exact solution to our problem in the case where the r.v. are $\eta(0,1)$. For $n = 2$ this probability is $1/\sqrt{2}$ (approximately 0.7071067810). This implies that the probability of obtaining a spiral (stable or unstable) is *exactly* $(1 - 1/\sqrt{2})/2$, or approximately 0.1464466095. As we have shown that the probability of obtaining a saddle is 0.5 we have that the probability of obtaining a node (stable or unstable) is *exactly* $(1/\sqrt{2} - 0.5)/2$, or approximately 0.1035533905. If we look at the simulation results for $n = 1.000.000$ we see that Monte-Carlo gave us a remarkably good approximation: 0.49961 for the saddle, 0.14656 for the stable spiral and 0.10351 for the stable node.

The reader who wants more details and the answers for the questions formulated in the beginning of this section should consult [3]. It can be found together with other interesting papers in the subject of random matrices at www-math.mit.edu/˜edelman/Edelman/publications.htm. Other fields where random matrices appear are number theory (for a connection to Riemann Hypothesis see [5]), numerical analysis (calculation of the condition number of a random matrix - see [2]) and multivariate statistics (test of hypothesis, principal component analysis, etc.).

Possible extensions of the present work include considering other symmetrical distributions or higher dimensional systems. In most cases it is difficult to obtain an exact solution as the one found in the 2×2 case with normally distributed r.v. In such cases one should appeal to simulation in order to obtain an approximation of the exact solution.

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