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Abstract. In this paper we study the dynamical formulation of the n-firm Cournot oligopoly model and variations. We consider both discrete and continuous cases and discuss its stability as well as its long-term behavior. As these resulting models are linear, we propose the use of techniques from linear algebra such as Shermann-Morrison's formula and Sylvester's Law of Inertia to deal with them. We describe an algorithm to approximate the eigenvalues and eigenvectors associated to the systems and illustrates the techniques with a simple example.
Keywords: ***oligopoly problem. eigenvalues. Shermann-Morrison formula.***

1 Introduction

In this work we apply known techniques of eigenvalue and eigenvector calculation to the Cournot oligopoly model. Our main goal is to study the general n-firm case and to provide efficient numerical methods to approximate the solution of the resulting systems. Other authors have already concerned about this more general setting. In [4], Theocharis gives the long term behavior of the dynamic Cournot model for a simple case, with an arbitrary number of firms. His conclusion is that for more than 3 firms the solution is always unstable. Fisher (see [2]) considered a wider class of models, which incorporates increasing marginal costs, and extends Theocharis results for the n firm model. He also provided sufficient conditions on the parameters of the model in order to guarantee the stability of the system.

In the next section we briefly describe Cournot's approach to the problem and derive the response curves for each firm. In section 3 we introduce time in the model in both discrete and continuous settings. Variations of the model such as quadratic production cost are also described. In section 4 we start with a simple case where an explicit answer is obtained, similar to [4]. Further, we introduce the mathematical formalism and necessary techniques to study stability and long-term behavior of these models. In section 5 we exemplify one of the techniques in a simple example and analyse its computational performance.

2 The classical Cournot model

The Cournot oligopoly model has been extensively studied. It is a model that considers n firms producing the same product competing for the market. The firms decide simultaneously the output and they all sell their product at the market clearing price. According to [5], the equations that define the model are

$$\begin{aligned} p &= K - Q \\ Q &= \sum_{i=1}^{n} q_i \\ C_i &= k_i q_i,\ i = 1, \ldots, n, \end{aligned}$$

where p is the common selling price, K is an arbitrary constant, q_i is the output of firm i, Q is the total amount of the product available in the market and C_i is the cost of production of firm i. One can define the production cost as any increasing function $f(q_i)$, such as q_i^2 or $\arctan(q_i)$ for example. We start using the linear function $f(q_i) = k_i q_i$ and later we consider

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the quadratic case. Note that the price equation is standard in supply demand models. The total revenue for each firm is pq_i and the profit π_i is given by

$$\pi_i = (K - Q)q_i - k_i q_i. \tag{1}$$

We define the Cournot equilibrium as the output level of each firm that maximizes its profit given the output of all other firms. In order to obtain these optimal quantities one must obtain the critical points of (1):

$$\frac{\partial \pi_i}{\partial q_i} = \left(K - \sum_{j=1}^{n} q_j \right) - q_i - k_i = 0$$

for each i. Solving for q_i we obtain

$$q_i = \frac{1}{2}\left(K - k_i - \sum_{j, j \neq i} q_j \right), \imath = 1, \ldots, n. \tag{2}$$

Each of these equations represent a hyperplane in the n-dimensional space. The Cournot solution of the system is the intersection point $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$.

Let us take a closer look at the Cournot equilibrium. It is an equilibrium in the sense that no firm would change its output knowing that its competitors will choose the prescribed output. This means that the market will sustain the equilibrium output levels indefinitely. The weakness of this equilibrium concept is that it does not tell us what happens if different output levels are observed. In the next section we will show how to introduce periods of time in the model in order to study the evolution of the system and its long term behavior given arbitrary initial output levels.

3 Dynamic models

In this section we describe the variations of the Cournot model based on [5]. There are two very different classes of dynamical models we will consider: discrete and continuous. These classes are defined by the way we model time, which can be counted in discrete periods or continuously. We first consider discrete models.

Under the discrete case we still have modelling issues. We have to specify in the model how future outputs depend on previous ones. In our formulation firms adjust their outputs using only other firms' output levels in the immediately previous period. Inspired by formula (2), the model can be written as

$$q_{i,t+1} = \frac{1}{2}\left(K - k_i - \sum_{j, j \neq i} q_{j,t} \right), \ i = 1, \ldots, n. \tag{3}$$

We refer to this model by the triplet (D, PM, L), where D stands for discrete, PM for profit maximization and L for linear costs. This model can be written in the form

$$q_{t+1} = Aq_t + b, \tag{4}$$

where $q_t = (q_{1,t}, \ldots, q_{n,t})$, A is a $n \times n$ matrix and b is a n-dimensional vector. Note that this is a system of n difference equations. It is easily seen that the matrix A is given by

$$A = \begin{pmatrix} 0 & -1/2 & \cdots & -1/2 \\ -1/2 & 0 & \cdots & -1/2 \\ \vdots & \vdots & \ddots & \vdots \\ -1/2 & -1/2 & \cdots & 0 \end{pmatrix}. \tag{5}$$

Another possibility is to diminish the discrepancy between the last output and the desired level (prescribed by formula (2))in the actual period:

$$q_{i,t+1} - q_{i,t} = \alpha_i (x_{i,t} - q_{i,t}), \ 0 < \alpha_i \leq 1, \ i = 1, \ldots, n, \tag{6}$$

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where $x_{i,t} = \frac{1}{2}(K - k_i - \sum_{j,j\neq i} q_{j,t})$. Notice that this model can be seen as a generalization of the previous one letting $\alpha = 1$. We refer to this model as (D,DD,L)[1]. This system can also be written in form (4).

In addition, we can consider quadratic costs instead of linear ones, that is, $C_i = k_i q_i^2$. This gives us two new models: (D,PM,Q)[2] and (D,DD,Q). Despite the presence of a quadratic term one can verify that both models can be written as in equation (4).

The other class of models considers continuous time. In this context we clearly cannot consider instantaneous adjustment. The first model (C,L) is constructed based on its discrete counterpart, but instead of difference equations we have a system of differential equations:

$$\frac{dq_i(t)}{dt} = \alpha_i(x_i(t) - q_i(t)),\ 0 < \alpha_i \leq 1,\ i = 1, \ldots, n,$$

where $q_i(t)$ is the output at time t and $x(t)$ is the continuous analog of Cournot solution in (2). The model can be written in the following general form:

$$\dot{q} = Aq + b, \tag{7}$$

where the dot denotes derivative with respect to t, A is again a $n \times n$ matrix and b is a n-dimensional vector. In the case (C,L) the matrix A is given by

$$A = \begin{pmatrix} -\alpha_1 & -\alpha_1/2 & \cdots & -\alpha_1/2 \\ -\alpha_2/2 & -\alpha_2 & \cdots & -\alpha_2/2 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_n/2 & -\alpha_n/2 & \cdots & -\alpha_n \end{pmatrix}. \tag{8}$$

All models we have shown lead to linear systems of equations and so, to study their stability, we have to find the spectrum, that is, the set of eigenvalues of the many matrices related to each of those systems. The presence of the vector b in the models does not influence the behavior of the system since one can get rid of it by a linear change of variables. We will get into this analysis in the next section.

4 Stability analysis

As we mentioned in the previous section, to obtain explicit solutions of the systems and to understand its long-term behavior all we need is to find the eigenvalues of the corresponding matrices. Moreover, certain conditions on the eigenvalues describe if the system will converge to the equilibrium or not. Mathematically, when we have a system of difference equations this condition is that all eigenvalues are inside the unitary disk (Figure 1) while in the continuous case one must have all the eigenvalues with negative real part (Figure 2). In this way, the problem reduces to efficient methods of calculating eigenvalues and that is the direction we will take now.

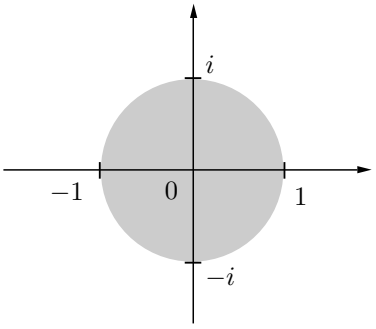

Figure 1: stability region for the discrete case.

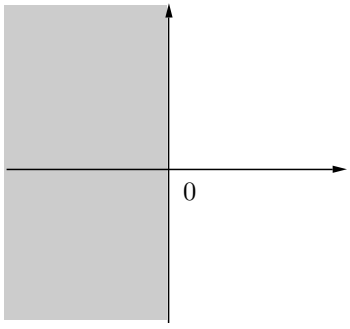

Figure 2: stability region for the continuous case.

Let us first analyse the (D,PM,L) case. A very similar model was studied in [4], so we just outline the eigenvalue calculation. Notice that the matrix A of this model (see (5)) can be decomposed in the sum $-1/2(U - I)$, where U is

[1] DD stands for diminished discrepancy.

[2] Q stands for quadratic.

the matrix whose entries are equal to 1 and I is the n-dimensional identity matrix. Once we know the spectrum of U, we can easily find the spectrum $\sigma(A)$ of A. If v is an eigenvector of U associated with the eigenvalue λ, we claim that v is an eigenvector of A with eigenvalue $-1/2(\lambda - 1)$. The proof is straightforward:

$$-1/2(U - I)v = -1/2(Uv - Iv) = -1/2(\lambda v - v) = -1/2(\lambda - 1)v. \tag{9}$$

The spectrum of the matrix U is elementary: first note that the matrix has rank 1 and therefore it has $n-1$ eigenvalues equals to zero. As the sum of the eigenvalues must be equal to the trace, we have that the missing eigenvalue is equal to n. Using the preceding calculation we conclude that $\sigma(A) = \{1/2, 1/2, \ldots, (1-n)/2\}$.

The result obtained implies that the long-term behavior depends solely on the number n of firms. If $n = 2$ the system is asymptotically stable, that is, it converges to the Cournot equilibrium q^*. If $n = 3$ the system exhibits oscillatory behavior. Finally, if $n \geq 4$ the system is unstable and the output levels become arbitrarily large in absolute value.

The matrix of the (D,PM,L) case is simple and we were able to explicitly calculate its eigenvalues. More sophisticated linear algebra tools are needed to calculate the eigenvalues of other models' matrices or at least to understand the long term behavior of each system.

(a) The signature of a matrix

An important concept in linear algebra is the signature of a matrix. It is well suited for our purposes, specially in the continuous case.

Definition 1 (Signature of a matrix) *Let A be a $n \times n$ symmetric matrix. Its* signature $\Sigma(A)$ *is the triplet formed by the the number of positive, negative and null eigenvalues of the matrix A.*

As an example note that the signature of the n-dimensional identity matrix is $(n, 0, 0)$, where the first coordinate represents the number of positive eigenvalues, the second the number of negative ones and the third the number of eigenvalues equals to zero.

We now state a classical theorem involving the signature of a matrix:

Theorem 2 (Sylvester's Law of Inertia) *Let A be a $n \times n$ symmetric matrix and let C be a non-singular $n \times n$ matrix. Then $\Sigma(A) = \Sigma(C^T AC)$.*

For an outline of the proof see ([6]). Remember from Figure 2 that the stability of a continuous model is determined by the sign of the real part of its eigenvalues. Even though the matrix of the (C,L) model is non-symmetric, we will see that its eigenvalues are real and then we can use Sylvester's Law to obtain its signature. We show how to proceed on a low-dimensional case only for clarity of exposition. One may directly extend the result for the general $n \times n$ case. Thus, recalling the (C,L) matrix defined in (8), we have for $n = 3$

$$A = \begin{pmatrix} -\alpha_1 & -\alpha_1/2 & -\alpha_1/2 \\ -\alpha_2/2 & -\alpha_2 & -\alpha_2/2 \\ -\alpha_3/2 & -\alpha_3/2 & -\alpha_3 \end{pmatrix}.$$

We note that studying $\Sigma(A)$ is the same as studying the signature of $B = -2A$: a positive eigenvalue of B represents a negative eigenvalue of A and vice-versa. We can rewrite B as a product of simpler matrices:

$$\begin{aligned} B &= \begin{pmatrix} 2\alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & 2\alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & 2\alpha_3 \end{pmatrix} = \begin{pmatrix} 2\alpha_1 & 0 & 0 \\ 0 & 2\alpha_2 & 0 \\ 0 & 0 & 2\alpha_3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2\alpha_1} & 0 & 0 \\ 0 & \sqrt{2\alpha_2} & 0 \\ 0 & 0 & \sqrt{2\alpha_3} \end{pmatrix} \begin{pmatrix} \sqrt{2\alpha_1} & 0 & 0 \\ 0 & \sqrt{2\alpha_2} & 0 \\ 0 & 0 & \sqrt{2\alpha_3} \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}. \end{aligned}$$

Using that for any two matrices P, Q we have $\sigma(PQ) = \sigma(QP)$, we see that

$$\begin{pmatrix} \sqrt{2\alpha_1} & 0 & 0 \\ 0 & \sqrt{2\alpha_2} & 0 \\ 0 & 0 & \sqrt{2\alpha_3} \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2\alpha_1} & 0 & 0 \\ 0 & \sqrt{2\alpha_2} & 0 \\ 0 & 0 & \sqrt{2\alpha_3} \end{pmatrix}$$

has the same spectrum of B. As the second matrix in this product is symmetric, we can use 2 to conclude that

$$S = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

has the same *signature* of B. This matrix can be written as $S = 1/2(U) + 1/2I$ and its spectrum is $\sigma(S) = \{1/2, 1/2, 2\}$, which can be easily calculated following the same steps as in the (D,PM,L) case. We observe that in the n-dimensional case we have $\sigma(S) = \{1/2, \ldots, 1/2, (n+1)/2\}$. Remembering that $B = -2A$, we see that $\Sigma(A) = (0, 3, 0)$ and hence the system is stable for any adjustment constants $\alpha_i,\ i = 1 \ldots 3$.

We note that Sylvester's Law of Inertia 2 is a limited tool in the sense that it does not explicitly calculate any eigenvalue of A. It solely gives us the signature of the matrix, which in the continuous case describes the stability, provided that the eigenvalues of the intermediate matrix (in our example it was matrix S) can be computed.

(b) Computing the eigenvalues explicitly

Now we study another classical method of linear algebra that will enable us to calculate the eigenvalues of the matrices involved in the models. The method was developed by Shermann and Morrison and it is a technique to solve linear systems of equations. A good reference for the method is [3].

Suppose we have a linear system $Ax = b$ where $A = B + u \otimes v$, B is a matrix whose inverse can be easily computed and $\otimes$ denotes tensor product, defined by $u \otimes v = uv^T$, where v^T is the transpose of the vector v .Denoting by $\langle u, v\rangle$ the inner product $u^T v$, one can proceed as follows:

$$\begin{aligned} Ax = b \quad &\Leftrightarrow \quad (B + u \otimes v)x = b \Leftrightarrow Bx + uv^T x = b \\ &\Leftrightarrow \quad Bx + u\langle v, x\rangle = b \Leftrightarrow x + B^{-1}u\langle v, x\rangle = B^{-1}b, \end{aligned} \tag{10}$$

where we pre-multiplied the last equality by B^{-1}. We consider x and $\langle v, x\rangle$ as two different terms and solve for $\langle v, x\rangle$. Then we take the inner product with v at both sides to get

$$\begin{aligned} &\langle v, x\rangle + \langle v, B^{-1}u\rangle\langle v, x\rangle = \langle v, B^{-1}b\rangle \\ &\Rightarrow \langle v, x\rangle = \frac{\langle v, B^{-1}b\rangle}{1 + \langle v, B^{-1}u\rangle}. \end{aligned}$$

Substituting $\langle v, x\rangle$ on the second equation of (10) we can write the solution as

$$x = B^{-1}b - \frac{B^{-1}u\langle v, B^{-1}b\rangle}{1 + \langle v, B^{-1}u\rangle}. \tag{11}$$

Recalling that the inverse matrix B can be easily computed, we have obtained a solution without inverting the original matrix A, which could be hard to calculate.

We are going to apply this method to obtain a numerical scheme to find all the eigenvalues of (D,PM,Q)-matrix below:

$$A = \begin{pmatrix} 0 & -\beta_1 & \cdots & -\beta_1 \\ -\beta_2 & 0 & \cdots & -\beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_n & -\beta_n & \cdots & 0 \end{pmatrix}, \tag{12}$$

where $\beta_i = 1/(2 + 2k_i)$. As before we go through the 3×3 case. We start by decomposing A as follows:

$$\begin{aligned} A &= \begin{pmatrix} 0 & -\beta_1 & -\beta_1 \\ -\beta_2 & 0 & -\beta_2 \\ -\beta_3 & -\beta_3 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_1} & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 \\ 0 & 0 & \sqrt{\beta_3} \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 \\ 0 & 0 & \sqrt{\beta_3} \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\ &\cong \begin{pmatrix} \sqrt{\beta_1} & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 \\ 0 & 0 & \sqrt{\beta_3} \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 \\ 0 & 0 & \sqrt{\beta_3} \end{pmatrix}, \end{aligned}$$

where $P \cong Q$ means that P and Q have the same eigenvalues. Note that this last product of matrices is a symmetric matrix itself, implying that all eigenvalues of A are real. Denoting by E the diagonal matrix with entries $E_{i,i} = \sqrt{\beta_i}$ and by $\mathbb{1}$ the vector $(1, \ldots, 1)^T$, we can write

$$A \cong E \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) E^T = E^2 - E\mathbb{1} \otimes E\mathbb{1}.$$

Note that matrix U, whose entries are equal to 1, is such that $U = \mathbb{1} \otimes \mathbb{1}$. In order to calculate the eigenvalues one has to consider the linear system

$$(E^2 - \lambda I - E\mathbb{1} \otimes E\mathbb{1})x = b, \tag{13}$$

where λ is a real number and b is an arbitrary vector with real entries. Remember from linear algebra that the values of λ for which the matrix $(E^2 - \lambda I - E\mathbb{1} \otimes E\mathbb{1})$ is singular are exactly the eigenvalues of $(E^2 - E\mathbb{1} \otimes E\mathbb{1})$. On the other hand, if λ is not in the spectrum of $(E^2 - E\mathbb{1} \otimes E\mathbb{1})$, then the matrix in (13) is invertible and we can use exactly the same procedure we outlined in the beginning of this section, with $u = v = E\mathbb{1}$ and $B = E^2 - \lambda I$. Running the calculations we get an expression similar to the one derived in (11):

$$\langle E\mathbb{1}, x \rangle = \frac{\langle E\mathbb{1}, (E^2 - \lambda I)^{-1} b \rangle}{1 - \langle E\mathbb{1}, (E^2 - \lambda I)^{-1} E\mathbb{1} \rangle}. \tag{14}$$

This expression is only valid if $\lambda \notin \sigma(E^2 - E\mathbb{1} \otimes E\mathbb{1})$ and $\lambda \neq \beta_i,\ i = 1, \ldots, 3$. Defining the denominator as a function of λ, $f(\lambda) = 1 - \langle E\mathbb{1}, (E^2 - \lambda I)^{-1} E\mathbb{1} \rangle$, we immediately identify the eigenvalues of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$ with the zeroes of $f(\lambda)$. Let us take a closer look at function $f(\lambda)$. Using that the inverse of a matrix M is equal to $M^{\text{cof}}\ (\det(M))^{-1}$, where M^{cof} is the cofactor matrix of M, we have

$$f(\lambda) = 1 - \left\langle E\mathbb{1}, \frac{(E^2 - \lambda I)^{\text{cof}}}{\det(E^2 - \lambda I)} E\mathbb{1} \right\rangle. \tag{15}$$

The inner product is clearly a quotient of two polynomials in λ and so is f(λ):

$$\frac{p(\lambda)}{q(\lambda)} = \left\langle E\mathbb{1}, \frac{(E^2 - \lambda I)^{\text{cof}}}{\det(E^2 - \lambda I)} E\mathbb{1} \right\rangle$$
$$\Rightarrow f(\lambda) = \frac{q(\lambda) - p(\lambda)}{q(\lambda)} = \frac{\tilde{p}(\lambda)}{q(\lambda)}.$$

The polynomial $q(\lambda)$ is the characteristic polynomial of E^2 and $\tilde{p}(\lambda)$ is the characteristic polynomial of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$ up to a constant C:

$$f(\lambda) = \mathrm{C} \frac{\det(E^2 - \lambda I - E\mathbb{1} \otimes E\mathbb{1})}{\det(E^2 - \lambda I)}. \tag{16}$$

This happens because $f(\lambda)$ and the characteristic polynomial of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$ have the same zeroes: the spectrum of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$. It is easy to see in (15) that this constant is equal to 1 due to the fact that the limit of $f(\lambda)$ is 1 when λ goes to infinity. From equation (15) it is possible to obtain a closed expression for $f(\lambda)$:

$$f(\lambda) = 1 - \sum_{i=1}^{n} \frac{\beta_i}{\beta_i - \lambda}. \tag{17}$$

In order to understand the graph of $f(\lambda)$ we will also determine its asymptotic behavior and the sign of its derivative. It is immediate to see that

$$\lim_{\lambda \to \beta_i^+} f(\lambda) = \infty \quad \text{and} \quad \lim_{\lambda \to \beta_i^-} f(\lambda) = -\infty.$$

The first derivative

$$f'(\lambda) = \sum_{i=1}^{n} \frac{\beta_i}{(\beta_i - \lambda)^2} < 0$$

is always negative for all real λ, which implies that we have exactly one eigenvalue between each pair $\beta_i, \beta_{i+1},\ i = 1, \ldots, n-1$, and one between $-\infty$ and β_1, assuming that the β's are indexed from the smallest to the largest and are different from each other. A sketch of function $f(\lambda)$ for $n = 3$ is outlined in Figure 4(b):

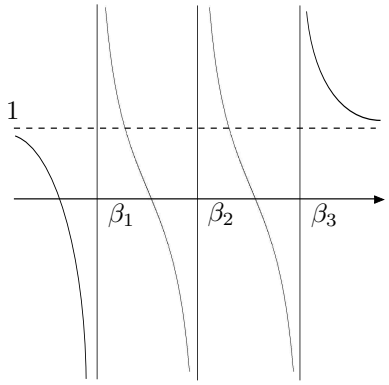

Figure 3: The graph of $f(\lambda)$

Let us summarize what we have done so far. We wanted to calculate the eigenvalues of matrix A in (12). We decomposed A in a product of simpler matrices whose spectrum is equal to $\sigma(A)$. Afterwards we identified this product as being a diagonal matrix E^2 minus a tensor product $E\mathbb{1} \otimes E\mathbb{1}$. If we set up the eigenvalue equation (13) for this matrix we fall in the case described by equation (10). Following the same steps we did for equation (10) we arrived at (14). Defining the denominator of (14) by $f(\lambda)$ we were able to conclude that the eigenvalues of the original matrix A were the zeroes of this function $f(\lambda)$.

Fortunately, not only the expression of $f(\lambda)$ is simple but also its zeroes are located between the asymptotes β_i, each one on an interval. To obtain a very good approximation of the eigenvalues of A one may use Newton's method at each interval, with initial condition given by, for instance, the mean value of the interval (except for the first interval, of course). In the next session we will run the calculations using this procedure for a very simple example.

One final remark: we have to be a little more careful when an eigenvalue of the diagonal matrix has multiplicity $k,\ k > 1$, that is, k of the diagonal entries are equal. What happens is that this eigenvalue is also an eigenvalue of the perturbation, but with multiplicity $k - 1$. We propose an intuitive explanation. Suppose two eigenvalues are very, very close. The theory developed in this section tells us that there is an eigenvalue of the perturbation between the asymptotes defined by the multiple eigenvalue and that the graph of $f(\lambda)$ between these two values is almost a vertical line. In the limit, the two asymptotes collapse and a multiple eigenvalue of the diagonal matrix arises, which is also an eigenvalue of the perturbation itself.

(c) Computing the eigenvectors

The hard work is to compute the approximate eigenvalues. Given these approximations, we present two straightforward procedures to approximate the eigenvalues with high accuracy. They both can be found in [1].

The first method is called *Inverse Iteration*. It is a variant of the well-known *Power Method*, where you arbitrarily choose a vector u and compute the sequence $u, Au, A^2u, \ldots$. It can be shown (see [1], page 154) that this sequence converges to the eigenvector with the largest absolute value. In our case we want not only the eigenvector associated with the largest eigenvalue but all the eigenvalues. To do so we need to slightly modify the power method: now we compute the power sequences with matrix $(A - \mu)^{-1}$ instead of A. Running the calculations for an arbitrary vector u, we obtain an approximation for the eigenvector associated with the eigenvalue closest to μ. As we have already approximated the eigenvalues of A, we just apply the inverse iteration with $\mu = \tilde{\lambda}$, that is, using matrix $(A - \tilde{\lambda})^{-1}$ for each approximated eigenvalue $\tilde{\lambda}$. With only a few iterations a very good approximation $\tilde{v_i}$ of the eigenvectors is obtained, which can be verified by computing $A\tilde{v_i} - \tilde{\lambda_i}\tilde{v_i}$ and observing how close to zero this quantity is.

Another procedure is based on lemma 5.2 of [1] and is specially designed for matrices of type $D + u \otimes u$, D diagonal, which is our case. The lemma states that if λ is an eigenvalue of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$, then $(E^2 - \lambda I)^{-1}E\mathbb{1}$ is the eigenvector associated to the eigenvalue λ. There is just one catch: we do not want an approximated eigenvector of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$ but from A, and we only know that these two matrices have the same spectrum. But we claim that if v is an eigenvector of $E^2 - E\mathbb{1} \otimes E\mathbb{1}$ associated to the eigenvalue λ, then Ev is an eigenvector of A. Indeed,

$$A(Ev) = E^2C(Ev) = E(ECE)v = E(E^2 - E\mathbb{1} \otimes E\mathbb{1})v = \lambda(Ev),$$

where C is the matrix $I - \mathbb{1} \otimes \mathbb{1}$. With this result one may easily obtain the approximated eigenvalue $\tilde{v}$ associated to the approximated eigenvector $\tilde{\lambda}$: first compute $\tilde{u} = (E - \tilde{\lambda}I)^{-1}E\mathbb{1}$ in order to obtain $\tilde{v} = E\tilde{u}$.

5 An example

We consider the (D,PM,Q) model with 4 firms competing for the market. Their cost coefficients are given in Table 1:

firm	cost coefficient
1	1.0
2	1.8
3	2.4
4	2.7

Table 1: Four firms and their cost coefficients.

The construction of matrix A for this example follows (12):

$$A = \begin{pmatrix} 0 & -.2500000000 & -.2500000000 & -.2500000000 \\ -.1785714286 & 0 & -.1785714286 & -.1785714286 \\ -.1470588235 & -.1470588235 & 0 & -.1470588235 \\ -.1351351351 & -.1351351351 & -.1351351351 & 0 \end{pmatrix}.$$

We now apply Newton's method to the function $f(\lambda)$ defined in (15). As we have shown, the zeroes of this function are the eigenvalues of matrix A. Remembering that the β_i's are always positive and equal to $1/((2+2k_i))$, one has to compute both $f(\lambda)$ and $f'(\lambda)$. We ran some experiments using Maple $9^{©}$ and chose as initial points the mid point of each interval and 0 for the interval $(-\infty, 0.1351351351)$. We used only three steps of the Newton iteration and the results are on Table 2.

Interval	Initial condition	Approximate eigenvalue
$(-\infty, .1351351351)$	0	-.4509528302
$(.1351351351, .1470588235)$	.1410969793	.1401195570
$(.1470588235, .1785714286)$	.1628151260	.1634805410
$(.1785714286, .2500000000)$	.2142857143	.2213393292

Table 2: Approximate eigenvalues of A.

In a larger dimensional problem we often do not know the true eigenvalues of matrix A. As this is an example and we want to check how accurate this procedure is, we numerically calculated the eigenvalues on Maple $9^{©}$ and the results are shown in Table 3.

Eigenvalues of A
$-.5249394270$
.1401195570
.1634805410
.2213393291

Table 3: Eigenvalues of A.

The first remark is that our scheme provided a remarkably good approximation for the eigenvalues of A with only three steps in Newton's methods. Of course, more steps in the algorithm provide a better approximation for the true eigenvalues of A. Also note that in this case all the eigenvalues lie on the interval $(-1, 1)$ and hence the system is stable.

One may argue that those initial conditions are artificial in the sense that for several choices of starting points we may fall outside the interval we started and miss the eigenvalue in that interval. However, numerical experiments suggest that the method is stable for almost every initial condition. To test this property we selected 100 equally spaced starting points in the interval $(.1351351351, .1470588235)$ and check their convergence to the eigenvalue $.1401195570$. The result we obtained was that all the approximations were correct to 8 decimal places with respect to the eigenvalue $.1401195570$, which lies in the interval.

To approximate the eigenvectors we will use the second approach described in section 4(c). Running the calculations we obtained the following approximations:

Approximate eigenvectors of A (normalized)
$(.6347521581, .4994282501, .4305810173, .4028164693)$
$(.0653833877, .1334572900, .6090112746, -.7791145178)$
$(.1822784368, .7464595362, -.5649130304, -.3007424142)$
$(.8727249132, -.4177512513, -.1980796926, -.1568424443)$

Table 4: Approximate eigenvectors of A.

To test the goodness of the approximation we calculated the differences $A\tilde{v} - \tilde{\lambda}\tilde{v}$ to see whether this values are close to zero. The results were remarkably good in the sense that all coordinates of the errors were very close to 0. Actually, they were less that 10^{-8}.

For the sake of comparison we also explicitly calculate the eigenvectors of matrix A on Maple $9^{\copyright}$ since the matrix in this example is only 4×4. Notice they are remarkably close to the approximations obtained in Table 5:

Eigenvectors of A (normalized)
$(.6347521581, .4994282501, .4305810173, .4028164694)$
$(-.0653833875, -.1334572894, -.6090112682, .7791145228)$
$(.1822784368, .7464595358, -.5649130312, -.3007424145)$
$(.8727249139, -.4177512510, -.1980796925, -.1568424442)$

Table 5: Eigenvectors of A

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6 Conclusion

In this paper we used classical linear algebra techniques to analyse stability of linear systems as well as explicitly calculate the eigenvalues of a matrix. Sylvester's Law of Inertia gives us a description of the qualitative behavior of a system without approximating its eigenvalues. This method is particular useful for continuous models whose stability is determined only by the signature of the system's matrix. On the other hand, Shermann-Morrison's formula lead us to an algorithm that not only describes the qualitative behavior but also approximates efficiently the spectrum of the subjacent matrix. The method reduces the calculus of the eigenvalues of the n-dimensional matrix to n one-dimensional Newton's method calculation, which is simpler to implement and gives excellent estimates for the true eigenvalues. Finally, once we approximated the eigenvalues we could approximate the respective eigenvectors using a formula given in [1].

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